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TAUBERIAN THEOREM OF LITTLEWOOD TYPE FOR SERIES IN BESSEL FUNCTIONS OF FIRST KIND[1]

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Abstract

A Tauberian theorem of Littlewood type for the summation of divergent series defined by means of Bessel functions of first kind is proved.

Key words and phrases: Bessel functions of first kind, Abelian and Tauberian theorems, summation of divergent series

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1. Introduction. A Tauberian theorem is a statement that relates the Abel summability and the standard convergency of a number series by means of some assumptions imposed on the general term of the series under question. In this paper we extend the validity of such type of assertion to series in Bessel functions.

Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

To define its Abel summability ([2], p. 20, 1.3 (2)), we also consider the power series $\sum\limits_{n=0}^{\infty} a_n z^n$.

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Definition 1.1. The series (1.1) is called A-summable if the series $\sum\limits_{n=0}^{\infty} a_n z^n$ converges in the disk $D = \{z : z \in \mathbb{C},\ |z| < 1\}$ and, moreover, there exists $\lim\limits_{z\to 1-0} \sum\limits_{n=0}^{\infty} a_n z^n = S$. The complex number S is called A-sum of the series (1.1) and the usual notation of that is

$$\sum_{n=0}^{\infty} a_n = S \quad (A).$$

Remark 1.1. *The A-summation is regular. It means that if the series* (1.1) *is convergent, then it is A-summable, and its A-sum is equal to its usual sum.*

Remark 1.2. *The A-summability of the series* (1.1) *does not imply in general its convergence. But, with additional conditions on the growth of the general term of the series* (1.1)*, the convergence can be ensured.*

A classical result in this direction is given by the following theorem ([2], Theorem 85).

Theorem 1.1 (Tauber)**.** *Let the series* (1.1) *be A-summable,*

$$\sum_{n=0}^{\infty} a_n = S \quad (A) \quad \textit{and} \lim_{n\to\infty} n a_n = 0.$$

Then the series $\sum\limits_{n=0}^{\infty} a_n$ *converges with a sum* S*.*

At first sight it seems that the condition $a_n = o(1/n)$ is essential. Nevertheless, Littlewood succeeds to weaken it and obtain the following strengthened version of the Tauber theorem ([6] 7.6.6; [2], Theorem 90).

Theorem 1.2 (Littlewood)**.** *Let the series* (1.1) *be A-summable,*

$$\sum_{n=0}^{\infty} a_n = S \quad (A) \quad \textit{and} \quad a_n = O(1/n).$$

Then the series $\sum\limits_{n=0}^{\infty} a_n$ *converges with a sum* S*.*

2. Series in Bessel functions of first kind. Let $J_n(z)$ ($z \in \mathbb{C}$, $n = 0, 1, 2, \dots$) be the Bessel functions of first kind. Consider a series of the form

$$\sum_{n=0}^{\infty} a_n J_n(z). \tag{2.1}$$

In studying the behaviour of such series on the boundary of its domain of convergence in the complex plane we recall that Cauchy-Hadamard and Abel type theorems are valid for such type of series as proven in [3].

Theorem 2.1 (of Cauchy-Hadamard type)**.** *The domain of convergence of the series* (2.1) *is the circular domain* $|z| < R$ *with a radius of convergence* $R = 1/\Lambda$*, where*

$$\Lambda = \frac{1}{2}\limsup_{n\to\infty}(|a_n|/n!)^{1/n}. \tag{2.2}$$

The cases $\Lambda = 0$ *and* $\Lambda = \infty$ *can be included in the general case, provided* $1/\Lambda$ *means* ∞*, respectively 0.*

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain where $2\varphi < \pi$ and vertex at the point $z = z_0$, symmetric with respect to the line through the points 0 and z_0.

Theorem 2.2 (of Abel type)**.** *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers, and* Λ *is defined by* (2.2), $0 < \Lambda < \infty$*. Let* $K = \{z : z \in \mathbb{C},\ |z| < R, R = 1/\Lambda\}$*. If* $f(z)$ *is the sum of the series* (2.1) *in the disk* K *and this series converges at the point* z_0 *of the boundary of* K*, then*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n J_n(z_0), \quad z \in g_\varphi, \quad |z| < R. \tag{2.3}$$

3. (J, z_0)-summation. Let $z_0 \in \mathbb{C}\setminus\mathbb{R}$ and $|z_0| = R$. Since all the zeros of $J_n(z)$ are real, then $J_n(z_0) \neq 0$. For the sake of brevity, denote

$$J_n^*(z; z_0) = \frac{J_n(z)}{J_n(z_0)}.$$

Definition 3.1. The series (1.1) is said to be (J, z_0)-summable if the series

$$\sum_{n=0}^{\infty} a_n J_n^*(z; z_0) \tag{3.1}$$

converges in the disk $|z| < R$ and, moreover, there exists the limit

$$\lim_{z\to z_0}\sum_{n=0}^{\infty} a_n J_n^*(z; z_0), \tag{3.2}$$

provided z remains on the segment $[0, z_0)$.

Remark 3.1. *Every* (J, z_0)*-summation is regular, and this property is just a particular case of Theorem 2.2.*

In [4] a Tauber type theorem for (J, z_0)-summation is given, namely:

Theorem 3.1 (of Tauber type)**.** *If the series* (1.1) *is* (J, z_0)*-summable and*

$$\lim_{n\to\infty} n a_n = 0, \tag{3.3}$$

then it is convergent.

4. An asymptotic formula. The asymptotic formula

$$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n (1+\theta_n(z)), \quad \theta_n(z) \to 0 \text{ as } n \to \infty \tag{4.1}$$

is given in ([7], § 17.81) for the Bessel coefficients. The convergence of $\{\theta_n(z)\}$ is uniform on the compact subsets of the complex plane $\mathbb{C}$. Considering explicitly $\theta_n(z)$, we can make this result sharper.

Theorem 4.1. *Let $K \subset \mathbb{C}$ be a nonempty compact set. Then there exists a constant C, $0 < C < \infty$, such that for each $n \in \mathbb{N}_0$ and each $z \in K$ the following inequality holds*

$$|\theta_n(z)| \leq C/(n+1). \tag{4.2}$$

Proof. First, let $z \in \mathbb{C}$. Due to (4.1) we can write

$$\theta_n(z) = \frac{1}{n+1}\sum_{m=1}^{\infty} \frac{(-1)^m (n+1)!}{m!(n+m)!}\left(\frac{z}{2}\right)^{2m}.$$

Denoting $u_m(z) = \dfrac{(-1)^m (n+1)!}{m!(n+m)!}\left(\dfrac{z}{2}\right)^{2m}$, we obtain the estimate

$$|u_m(z)| \leq \frac{1}{m!}\left|\frac{z}{2}\right|^{2m} \tag{4.3}$$

for the absolute value of $u_m(z)$. The series

$$\sum_{m=1}^{\infty} \frac{1}{m!}\left|\frac{z}{2}\right|^{2m} \tag{4.4}$$

converges for each $z \in \mathbb{C}$ and its sum is $\exp(|z|^2/4) - 1$. This shows that

$$|\theta_n(z)| \leq \frac{1}{n+1}\left(\exp(|z|^2/4) - 1\right) \tag{4.5}$$

on the whole complex plane. Then, the estimate (4.2) follows immediately from (4.5). □

Remark 4.1. *The uniform convergence of $\theta_n(z)$ on the compact subsets of $\mathbb{C}$ follows from* (4.2) *as well.*

Remark 4.2. *Formula* (4.1) *has been used in the proof of Theorems* 2.1, 2.2, 3.1. *We apply* (4.2) *essentially in proving the corresponding strengthened version of Tauber theorem for series in Bessel functions of first kind.*

5. A Littlewood type theorem. A Littlewood generalization of the $o(n)$ version of Tauber type theorem (Theorem 3.1) is given in this part. Similar theorem is proved in [1] (a generalization of a Tauber type theorem, proved in [5]) for summation by means of Laguerre polynomials.

Theorem 5.1 (of Littlewood type)**.** *If the series* (1.1) *is* (J, z_0)*-summable and*

$$a_n = O(1/n) \tag{5.1}$$

then the series (1.1) *converges.*

Proof. Let z belong to the segment $[0, z_0]$. By using the asymptotic formula (4.1) for the Bessel functions of first kind, we obtain

$$a_n J_n^*(z; z_0) = a_n \left(\frac{z}{z_0}\right)^n \frac{1 + \theta_n(z)}{1 + \theta_n(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_n(z; z_0)\right),$$

where $\widetilde{\theta}_n(z; z_0) = \dfrac{\theta_n(z) - \theta_n(z_0)}{1 + \theta_n(z_0)}$. Then $\widetilde{\theta}_n(z; z_0) = O(1/n)$, due to (4.2).

Let us write (3.1) in the form

$$\sum_{n=0}^{\infty} a_n J_n^*(z; z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_n(z; z_0)\right). \tag{5.2}$$

Denoting $w_n(z) = a_n \left(\dfrac{z}{z_0}\right)^n \widetilde{\theta}_n(z; z_0)$ we consider the series $\sum\limits_{n=0}^{\infty} w_n(z)$. Since $|w_n(z)| \le |a_n| \, |\widetilde{\theta}_n(z; z_0)|$ and according to condition (5.1) and Theorem 4.1, there exists a constant C, such that $|w_n(z)| \le C/n^2$. Since $\sum\limits_{n=1}^{\infty} 1/n^2$ converges, the series $\sum\limits_{n=0}^{\infty} w_n(z)$ is also convergent, even absolutely and uniformly on the segment $[0, z_0]$. Therefore (since $\lim\limits_{z \to z_0} w_n(z) = 0$)

$$\lim_{z \to z_0} \sum_{n=0}^{\infty} w_n(z) = \sum_{n=0}^{\infty} \lim_{z \to z_0} w_n(z) = 0. \tag{5.3}$$

Obviously, the assumption that the series (1.1) is (J, z_0)-summable implies the existence of the limit (3.2). Then, having in mind that (5.2) can be written in the form

$$\sum_{n=0}^{\infty} a_n J_n^*(z; z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n + \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \widetilde{\theta}_n(z; z_0),$$

we conclude that there exists the limit

$$\lim_{z \to z_0} \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \tag{5.4}$$

and, moreover,

$$\lim_{z\to z_0}\sum_{n=0}^{\infty} a_n J_n^*(z;z_0) = \lim_{z\to z_0}\sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n . \tag{5.5}$$

From the existence of the limit (5.4) it follows that the series (1.1) is A-summable. Then according to Theorem 1.2, the series (1.1) converges. □

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